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Approximation With Constraints In Normed Linear Spaces

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1. INTRODUCTION

The purpose of this paper is to develop a unified approach to the characterization of solutions of constrained and unconstrained approximation problems. Several papers have been written on the characterization of solutions of special approximation problems with particular types of constraints or without constraints. For uniform approximation a general theory has been obtained by using generalized weight functions. Recently a new approach via optimization theory has been presented in [1]. The idea is to show, first, that the local Kolmogoroff condition is satisfied. Assuming a convexity condition, it can be shown that the local Kolmogoroff condition implies the Kolmogoroff criterion. Hence best approximations are characterized by the local Kolmogoroff condition.

An essential restriction in [1] is the assumption of linear equality constraints For uniform approximation problems with nonlinear equality constraints, the local Kolmogoroff condition has been deduced in [2] under the assumption of a regularity condition that does not seem to be practical. By deleting inequality constraints a more satisfactory regularity condition has been studied in [3].

Our aim is to treat approximation problems with nonlinear equality and inequality constraints in a normed linear space and to present a new and satisfactory regularity condition. As in [1], we consider the problem as a particular type of optimization problem.

Applying new kinds of differentiability, a new approach to optimization problems has been developed in [4]. A generalization of the well-known Lagrange multiplier theorem has been obtained that can be applied to convex optimization problems as well as to differentiable optimization problems. Here we shall apply this theorem to approximation problems with constraints. In particular we obtain new characterization theorems for constrained L_p-approximation problems of continuous functions.

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2. The Maximum Principle

We assume that I is a finite set, Γ_j $(j \in I)$ are compact topological spaces, E and Z are Banach spaces, and M_0 is an open subset of E. Let the mappings $g_{\tau,j} : E \to \mathbb{R}$ $(\tau \in \Gamma_j, j \in I)$, p: $E \to Z$, and f: $E \to \mathbb{R}$ be given. The problem is to characterize a (local) minimum of f on the set

$$M = \bigcap_{j\in I} \bigcap_{\tau\in\Gamma_j} \{x \in E: g_{\tau,j}(x) \leqslant 0\} \cap \{x \in E: p(x) = \Theta\} \cap M_0 .$$

Let $G_j(x) = \max\{g_{\tau,j}(x) : \tau \in \Gamma_j\}$ $(x \in E, j \in I)$. Then the problem is equivalent to: find a (local) minimum of f on the set $\{x \in M_0 : G_j(x) \leqslant 0\ (j \in I), p(x) = \Theta\}$

Let $\tau > 0$. The set of mappings r: $(0, \tau] \to E$ so that $\lim_{t\to 0+} (r(t)/t) = \Theta$ is denoted by $H_\tau(E)$ and $H(E) = \bigcup_{\tau>0} H_\tau(E)$.

Definition. Let E_1 and E_2 be normed linear spaces and $x_0 \in E_1$.

(a) Let $C \subset E_2$ and γ: $E_1 \to E_2$ a mapping. A mapping $\gamma'(x_0)$: $E_1 \to E_2$ is called an $H(E_1)$-variation of γ at x_0 with respect to C, if $h \in E$, $\gamma'(x_0)\, h \in C$ and $r \in H(E_1)$ imply the existence of a $\tau > 0$ so that $(1/t)(\gamma(x_0 + th + r(t)) - \gamma(x_0)) \in C$ for every $t \in (0, \tau]$.

(b) The mapping γ: $E_1 \to E_2$ is called G-differentiable at x_0, if there is a continuous, linear mapping $\gamma'(x_0)$: $E_1 \to E_2$ so that for every $h \in E$ $\lim_{t\to 0+}(1/t)(\gamma(x_0 + th) - \gamma(x_0)) = \gamma'(x_0)h$.

(c) Let Γ be a compact topological space. A set of mappings γ_τ : $E_1 \to E_2$ $(\tau \in \Gamma)$ has property:

(1) (D) at x_0, if there is a neighborhood U of x_0 so that the mapping $\tau, x \mapsto \gamma_\tau(x)$ $(\Gamma \times U \to E_2)$ is continuous.

(2) (D1) at x_0, if

(i) it has property (D) at x_0, and

(ii) there is a neighborhood U of x_0 so that for every $\tau \in \Gamma$ γ_τ is G-differentiable at every $x \in U$ and the mapping $(\tau, x, h) \mapsto \gamma_\tau'(x)h$ $(\Gamma \times U \times E_1 \to E_2)$ is continuous.

(d) A mapping γ: $E_1 \to E_2$ has property (D2) at x_0, if

(i) γ is continuous at x_0

(ii) there is a neighborhood U of x_0 so that γ is G-differentiable at every $x \in U$ and the mapping $x \mapsto \gamma'(x)$ $(U \to E^*)$ is continuous. E^* denotes the normed, linear space of continuous, linear functionals on E.

Let us introduce the notation for $x_0 \in M$

$$\Gamma_j(x_0) = \{\tau \in \Gamma_j : g_{\tau,j}(x_0) = 0\} \qquad (x \in E, j \in I),$$
$$I_0 = \{j \in I : \Gamma_j(x_0) \neq \varnothing\},$$

and $G_j'(x_0)h = \max\{g'_{\tau,j}(x_0)h : \tau \in \Gamma_j(x_0)\}$ $(j \in I_0,\ x \in U,\ h \in E)$ if $(g_{\tau,j})_{\tau\in\Gamma_j}$ has property (D1) at x_0.

LEMMA 1. *Let $j \in I_0$ and $x_0 \in M$. Suppose $(g_{\tau,j})_{\tau\in\Gamma_j}$ has property* (D1) *at x_0. If $h \in E$ and $G_j'(x_0)h < 0$, then for every $r \in H(E)$ there is a $\tilde{\tau} > 0$ so that $G_j(x_0 + th + r(t)) < 0$ for every $t \in (0, \tilde{\tau}]$.*

Proof. Let $\tilde{h} \in E$ with $G_j'(x_0)\tilde{h} < 0$,

$$\delta = \max\{g'_{\tau,j}(x_0)\,\tilde{h}: \tau \in \Gamma_j(x_0)\} < 0.$$

For brevity we now omit the subscript j.

There is a neighborhood U of x_0 so that the mapping $(\tau, x, h) \to g_\tau'(x)h$ $(\Gamma \times U \times E \to \mathbb{R})$ is continuous. Thus there are an open neighborhood Γ_0 of $\Gamma(x_0)$, an open, convex neighborhood U_0 of x_0, and an open neighborhood V_0 of $\tilde{h}$ so that $g_\tau'(x)h < \delta/2$ for every $\tau \in \Gamma_0$, $x \in U_0$, and $h \in V_0$. Since $\tau \notin \Gamma_0 \Rightarrow g_\tau(x_0) < 0$ and since $\Gamma\backslash\Gamma_0$ is compact, we have

$$\mu = \max\{g_\tau(x_0) \mid \tau \in \Gamma\backslash\Gamma_0\} < 0$$

if $\Gamma_1 = \Gamma\backslash\Gamma_0 \neq \varnothing$.

Let $r \in H(E)$. There is a $t_1 > 0$ so that $x_0 + t\tilde{h} + r(t) \in U_0$ and $\tilde{h} + (r(t)/t) \in V_0$ for every $t \in (0, t_1]$. If $\Gamma_1 \neq \varnothing$ then choose $t_2 \in (0, t_1]$ so that

$$t_2 \sup \left\{\left| g_\tau'(x_0 + \theta t\tilde{h} + \theta r(t)) \left(\tilde{h} + \frac{r(t)}{t}\right)\right| \,\middle|\, \tau \in \Gamma_1,\right.$$
$$\left. t \in (0, t_1],\ \theta \in [0, 1]\right\} < -\mu.$$

By the mean value theorem we obtain

$$g_\tau(x_0 + t\tilde{h} + r(t)) - g_\tau(x_0) = tg_\tau'(x_0 + \theta t\tilde{h} + \theta r(t)) \left(\tilde{h} + \frac{r(t)}{t}\right).$$

Thus

$$g_\tau(x_0 + t\tilde{h} + r(t)) - g_\tau(x_0) < t\delta/2 < 0$$

for every $\tau \in \Gamma_0$ and $t \in (0, t_1]$.

If $\Gamma = \Gamma_0$ then $g_\tau(x_0 + t\tilde{h} + r(t)) < 0$ for every $\tau \in \Gamma$ and $t \in (0, t_1]$ and the proof is complete. Now suppose $\Gamma_0 \neq \Gamma$. For $\tau \in \Gamma\backslash\Gamma_0$ and $t \in (0, t_2]$ we obtain

$$g_\tau(x_0 + t\tilde{h} + r(t)) \leqslant tg_\tau'(x_0 + \theta t\tilde{h} + \theta r(t)) \left(\tilde{h} + \frac{r(t)}{t}\right) + \mu$$
$$\leqslant \mu + t_2 \left| g_\tau'(x_0 + \theta t\tilde{h} + \theta r(t)) \left(\tilde{h} + \frac{r(t)}{t}\right)\right| < 0,$$

hence $g_\tau(x_0 + t\tilde{h} + r(t)) < 0$ for every $\tau \in \Gamma$ and $t \in (0, t_2]$.

COROLLARY 2. *If the conditions of Lemma* 1 *are satisfied, then* $G_j'(x_0)$ *is a* $H(E)$*-variation of* G_j *at* x_0 *with respect to* $(-\infty, 0)$.

Now we apply Theorem II, 2.4 of [4, Maximumprinzip] and obtain
Theorem 3. Let $x_0 \in M$. Assume that the sets $(g_{\tau,j})_{\tau\in\Gamma_j}$ $(j \in I_0)$ have property (D1) at x_0, p has property (D2) at x_0, and the sets $(g_{\tau,j})_{\tau\in\Gamma_j}$ $(j \in I\setminus I_0)$ have property (D) at x_0. If x_0 is a local minimum of f on M, and there is a convex $H(E)$-variation $f'(x_0)$ of f at x_0 with respect to $(-\infty, 0)$, then there are numbers $\hat{l} \geqslant 0$, $l_j \geqslant 0$ $(j \in I_0)$, and a linear functional ϕ on Z so that

$$\hat{l}f'(x_0)\,h + \sum_{j\in I_0} l_j G_j'(x_0)\,h + \phi \circ p'(x_0)\,h \geqslant 0$$

for every $h \in E$; at least one number $\hat{l}$ or l_j or the functional ϕ is different from 0 or Θ, respectively.

If $p'(x_0)$ is surjective and there is an $h_0 \in E$ so that $G_j'(x_0)h_0 < 0$ for every $j \in I_0$ and $p'(x_0)h_0 = \Theta$, then $\hat{l} > 0$. We call x_0 regular if this condition holds. M is regular if every $x \in M$ is regular.

3. THE LOCAL KOLMOGOROFF CONDITION

Suppose T is a normed linear space and V is a subset of T. The set of best approximations to $w \in T$ with respect to V is the set

$$P[w, V] = \{v_0 \in V : \text{for every } v \in V\ \|w - v\| \geqslant \|w - v_0\|\}.$$

Let Σ_{v_0} be the set of linear functionals l on T such that $l(w - v_0) = \|w - v_0\|$ and $|l(z)| \leqslant \|z\|$ for every $z \in T$. The global Kolmogoroff criterion is. If V is convex then $v_0 \in P[w, V]$ if and only if

$$\min\{l(v - v_0) \in \mathbb{R}: l \in \Sigma_{w-v_0}\} \leqslant 0$$

for every $v \in V$. For an elementary proof without using extremal functionals see [4].

Let E be a Banach space and let $F: E \to T$ be a mapping. We consider the approximation by elements of the set $V = F[M]$ where M is defined as in Section 2. In this Section we assume that $x_0 \in M$, F is Fréchet differentiable at x_0, the sets $(g_{\tau,j})_{\tau\in\Gamma_j}$ and the mapping p satisfy the conditions of Theorem 3. Let $f: E \to \mathbb{R}$ be defined by $f(x) = \|w - F(x)\|$ $(x \in E)$. Then $v_0 = F_0(x_0)$ is a best approximation to w with respect to $F[M]$, if and only if x_0 is a minimum of f on M. Applying [4, Lemma II, 3.2 and Lemma II, 4.5] we obtain

LEMMA 4. *The mapping* $f'(x_0)$: $E \to \mathbb{R}$ *defined by*

$$f'(x_0)h = \|w - F(x_0) - F'(x_0)h\| - \|w - F(x_0)\| \qquad (h \in E)$$

is a convex $H(E)$*-variation of* f *at* x_0 *with respect to* $(-\infty, 0)$.

Let $E(x_0) = \{h \in E : p'(x_0)h = \Theta,\ G_j'(x_0)h \leqslant 0\ (j \in I_0)\}$. $F'(x_0)\ [E(x_0)]$ is convex. Thus Corollary 2 and Theorem 3 imply

THEOREM 5. *If $v_0 = F(x_0)$ is a best approximation to w and x_0 is regular, then $\Theta \in P[w - F(x_0), F'(x_0)\ [E(x_0)]]$ and for every $h \in E(x_0)$*

$$\min\{l \circ F'(x_0)\ h\colon l \in \Sigma_{w-v_0}\} \leqslant 0$$

(local Kolmogoroff condition).

If we particulaire Theorem 5 to the uniform approximation of continuous functions, we obtain a result of Hoffmann [2]; in view of Ljusternik's theorem (see [4]) our regularity condition implies the condition used in [2] so it is more restrictive, but it seems to be easier to apply.

We point out that the general theory developed in [4] is as well applicable to approximation problems with asymmetric norms as the one used in [5].

4. THE Θ OF CONVEX HULL THEOREM

In addition to the assumptions of Section 3 we suppose that there are elements $e_1,..., e_n \in E$ with $\text{span}\{F'(x_0)\, e_1,..., F'(x_0)e_n\} = F'(x_0)\ [P(x_0)]$ where $P(x_0) = \{h \in E : p'(x_0)h = \Theta\}$. Let $A_j(x_0)$ be the set of vectors $-(g'_{\tau,j}(x_0)e_1,..., g'_{\tau,j}(x_0)e_n)$ with $\tau \in \Gamma_j(x_0)$ and $j \in I_0$ and $A(x_0)$ the set of vectors $(l \cdot F'(x_0)e_1,..., l \cdot F'(x_0)e_n)$ with $l \in \Sigma_{w-v_0}$. Suppose that $\Theta \notin \text{con}\{A(x_0) \cup \bigcup_{j\in I_0} A_j(x_0)\}$. Applying the separation theorem there is an $h \in P(x_0)$ so that $l \cdot F'(x_0)h > 0$ for every $l \in \Sigma_{w-v_0}$ and $g'_{\tau,j}(x_0)h < 0$ for every $\tau \in \Gamma_j(x_0)$ and $j \in I_0$. Hence $h \in E(x_0)$ which contradicts Theorem 5 if x_0 is regular. But if x_0 is regular then $\Theta \notin \text{con}\{\bigcup_{j\in I_0} A_j(x_0)\}$. So we have proved.

Theorem 6. If $v_0 = F(x_0)$ is a best approximation to w and x_0 is regular, then Θ can be written as a convex combination of elements of the set $A(x_0) \cup \bigcup_{j\in I_0} A_j(x_0)$ with at least one point from the set $A(x_0)$ included nontrivially.

5. REGULARITY

In this section it will be shown that many constrained approximation problems are regular.

EXAMPLE 1. Linear equality or inequality constraints on parameters [6–9].

(a) $E = \mathbb{R}^n$, $Z = \mathbb{R}^m$, $M = \{x \in E : Ax = b\}$, A is a real $m \times n$-matrix, $b \in \mathbb{R}^m$. M is regular if A is surjective (rank $A = m$).

(b) $E = \mathbb{R}^n$, $J_1 \subset \{1,..., n\}$, $J_2 \subset \{1,..., n\}$, a_j $(j \in J_2)$ are positive real numbers, $M = \{x \in E : x_j \geqslant 0 \text{ if } j \in J_1,\ |x_j| \leqslant a_j \text{ if } j \in J_2\}$. M is regular.

EXAMPLE 2. Monotone approximation [10]. $K \in \mathbb{N}_0$, $n \in \mathbb{N}$, $v_1, \ldots, v_n \in C^K[0, 1]$, $I = \{j \in \mathbb{Z} : 1 \leqslant j \leqslant K + 1\}$. Let $\Gamma_1, \ldots, \Gamma_{K+1}$ be closed subsets of $[0, 1]$ and $\epsilon_0, \ldots, \epsilon_K \in \{-1, 1\}$. Suppose $u_1, \ldots, u_K \in C[0, 1]$ are given; $E = \mathbb{R}^n$:

$$g_{\tau,j}(x_1, \ldots, x_n) = \epsilon_{j-1} \left[\sum_1^n x_\mu v_\mu^{(j-1)}(\tau) - u_{j-1}(\tau) \right] \qquad (\tau \in \Gamma_j, j \in I, x \in E.)$$

(a) $K = 0$, span$\{v_1, \ldots, v_n\}$ is a Haar space. Then there is a $\hat{v} \in$ span $\{v_1, \ldots, v_n\}$ so that $\epsilon_0 \hat{v}(t) < 0$ for every $t \in [0, 1]$. Hence M is regular.

(b) $K \geqslant 1$, span $\{v_1, \ldots, v_n\} = P^{n-1}$ (polynomials of degree $\leqslant n - 1$).

Then there is a $\hat{v} \in P^{n-1}$ so that $\epsilon_j \hat{v}^{(j)}(t) < 0$ for every $t \in [0, 1]$ and $j = 0, 1, \ldots, K$ (see [10]). Hence M is regular.

EXAMPLE 3. Restricted range approximation [5, 11–15]. $n \in \mathbb{N}$, $v_1, \ldots, v_n \in C[0, 1]$ linear independent. Let Γ be a closed subset of $[0, 1]$, $E = \mathbb{R}^n$, $I = \{1, 2\}$, and $l, u \in C[0, 1]$:

$$g_{\tau,1}(x_1, \ldots, x_n) = \sum_1^n x_\mu v_\mu(\tau) - u(\tau) \qquad (\tau \in \Gamma, x \in E),$$

$$g_{\tau,2}(x_1, \ldots, x_n) = l(\tau) - \sum_1^n x_\mu v_\mu(\tau) \qquad (\tau \in \Gamma, x \in E).$$

Suppose $l(\tau) < u(\tau)$ for every $\tau \in \Gamma$ and assume that at least two points x, $\bar{x} \in M$ are given so that $\sum_1^n (x_\mu - \bar{x}_\mu) v_\mu \neq \Theta$. Let $V = \text{span}\{v_1, \ldots, v_n\}$. For $v \in V$ the mapping $\sigma_v : \Gamma \to \{-1, 0, 1\}$ is given by

$$\sigma_v(\tau) \begin{cases} = 1 & \text{if } \; v(\tau) = u(\tau) \\ = -1 & \text{if } \; v(\tau) = l(\tau) \\ = 0 & \text{if } \; l(\tau) < v(\tau) < u(\tau). \end{cases}$$

Suppose V is a Haar space. Let $v \in V$, $l(\tau) \leqslant v(\tau) \leqslant u(\tau)$ for every $\tau \in \Gamma$ and $0 \leqslant \tau_0 < \cdots < \tau_n \leqslant 1$. Then there is a $\hat{\jmath} \in \{0, 1, \ldots, n-1\}$ so that $\sigma_v(\tau_{\hat{\jmath}}) \, \sigma_v(\tau_{\hat{\jmath}+1}) \neq -1$. That means v "alternates" at most n times. Then there is a $\hat{v} \in V$ so that $\hat{v}(\tau) < 0$ if $v(\tau) = u(\tau)$ and $\hat{v}(\tau) > 0$ if $v(\tau) = l(\tau)$ (use a theorem of Krein (see [26]) on polynomials of Haar spaces with prescribed zeros). Then M is regular.

A similar argument shows that the problem is regular if rational restricted range approximation is considered. If $l(\tau) \leqslant u(\tau)$ is assumed as in [16–18] we do not know if the problem remains regular.

EXAMPLE 4. Interpolatory constraints [3, 19–24].

When considering nonlinear equality constraints only our regularity condition is more general than the condition used in [3]. Hence problems with interpolatory constraints, as considered in [3], are regular.

EXAMPLE 5. Interpolatory constraints and inequality constraints on parameters.

Let $n \in \mathbb{N}$, $r \in \mathbb{N}$. $E = \mathbb{R}^n$, $Z = \mathbb{R}^r$, $u_1, \ldots, u_n \in C[0, 1]$ linear independent so that $\text{span}\{u_1, \ldots, u_n\}$ is a Haar space, $0 \leqslant t_1 < \cdots < t_r \leqslant 1$; $\sigma_1, \ldots, \sigma_r \in \mathbb{R}$. Let $g_j(x_1, \ldots, x_n) = -x_j$ ($j \in \{1, \ldots, n\}, x \in E$) $p(x_1, \ldots, x_n) = (\sum x_j u_j(t_1) - \sigma_1, \ldots, \sum x_j u_j(t_r) - \sigma_r)$ $(x \in E)$. Let $J \subset \{1, \ldots, n\}$ so that $x \in M$ $j \notin J$ imply $x_j = 0$ and there is a $\bar{x} \in M$ so that $\bar{x}_j > 0$ for every $j \in J$.

Let $x_0 \in M$ and $h = \bar{x} - x_0$. $p'(x_0)$ is surjective, $p'(x_0)\, h = \Theta$, $g_j'(x_0)h = -h_j = -\bar{x}_j < 0$ if $j \in I_0$. Hence M is regular.

6. CHARACTERIZATION OF SOLUTIONS

If $F[M]$ is an α-sun(see[25]), then best approximations are characterized by the Kolmogoroff criterion. A sufficient condition for $F[M]$ to be an α-sun is a property, we call it property S, used in [25]. If this condition is satisfied, then best approximations are characterized by the local Kolmogoroff condition.

DEFINITION. (F, M) has property S, if the following holds. Suppose v, $v_0 \in F[M]$, $v \neq v_0$, $w \in T$, and $l(v - v_0) > 0$ for every $l \in \Sigma_{w-v_0}$. Then there are $x_0 \in M$, $h \in E(x_0)$, so that $F(x_0) = v_0$ and $l \cdot F'(x_0)h > 0$ for every $l \in \Sigma_{w-v_0}$.

If F is linear, p and $g_{\tau,j}$ $(\tau \in \Gamma, j \in I)$ are linear, then (F, M) has property S. In [1–3, 5, 9–11, 13, 14, 19, 20, 24] linear uniform approximation problems with linear constraints are studied. For those problems regularity has been proved in Section 5. Since in these problems (F, M) has property S, best approximations are characterized by the local Kolmogoroff condition. Our result implies furthermore that for linear L_p-approximation problems with linear constraints this statement equally holds. We drop the detailed presentation of these results.

7. APPLICATIONS TO RATIONAL APPROXIMATION

Let n, $m \in \mathbb{N}$. Let $u_1, \ldots, u_n \in C[0, 1]$ and $v_1, \ldots, v_m \in C[0, 1]$ be two sets of linear independent functions:

$$N = \left\{ y \in \mathbb{R}^m \colon \sum_1^m y_j v_j(t) > 0 \text{ for every } t \in [0, 1] \right\},$$

$E = \mathbb{R}^n \times \mathbb{R}^m$, $M_0 = \mathbb{R}^n \times N$. Let the mappings $P: \mathbb{R}^n \to C[0, 1]$, $Q: \mathbb{R}^m \to C[0, 1]$, and $F: M_0 \to C[0, 1]$ be defined by

$$P(x) = \sum_1^n x_j u_j \qquad (x \in \mathbb{R}^n),$$

$$Q(y) = \sum_1^m y_j v_j \qquad (y \in \mathbb{R}^m),$$

$$F(x, y) = P(x)/Q(y) \qquad ((x, y) \in M_0).$$

We consider best uniform approximations of continuous functions from $F[M]$, where M is some subset of M_0.

(a) Suppose Γ is a compact subset of $[0, 1]$. Let $l, u \in C[0, 1]$ so that $l(\tau) < u(\tau)$ for every $\tau \in \Gamma$.

$$g_{\tau,1}(x, y) = F(x, y)(\tau) - u(\tau) \qquad (\tau \in \Gamma, (x, y) \in M_0),$$
$$g_{\tau,2}(x, y) = -F(x, y)(\tau) + l(\tau) \qquad (\tau \in \Gamma, (x, y) \in M_0),$$
$$M = \{(x, y) \in M_0 : g_{\tau,1}(x, y) \leqslant 0, g_{\tau,2}(x, y) \leqslant 0 \text{ for every } \tau \in \Gamma\}.$$

(b) Let R and S be $k \times n$ and $k \times m$-matrices so that $\operatorname{rank}(R, S) = k$.

$$p(x, y) = Rx + Sy \qquad ((x, y) \in E,$$
$$M = \{(x, y) \in E : p(x, y) = \Theta\}.$$

(c) Let $I \subset \{1, ..., n\}$.

$$g_j(x, y) = -x_j \qquad (j \in I, (x, y) \in E),$$
$$M = \{(x, y) \in E : g_j(x, y) \leqslant 0 \text{ for every } j \in I\}.$$

Let us extend F and $g_{\tau,j}$ ($\tau \in \Gamma, j \in \{1, 2\}$) to E. As we have seen in Section 5 M is regular if it is defined as in (a), (b), and (c).

For $(x, y) \in M_0$ let

$$T_0(x, y) = \operatorname{span}\{u_1, ..., u_n, -F(x, y)v_1, ..., -F(x, y)v_m\}$$

and

$$T(x, y) = \{h/Q(y) : h \in T_0(x, y)\}.$$

Then for $(x, y) \in M$ $F'(x, y)$ $[E(x, y)]$ is in:

Case (a), the set of functions $q \in T(x, y)$ so that $q(\tau) \leqslant 0$ if

$$F(x, y)(\tau) = u(\tau) \qquad \text{and} \qquad q(\tau) \geqslant 0 \qquad \text{if} \quad F(x, y)(\tau) = l(\tau).$$

Case (b), the set of functions $q = (\sum a_j u_j - F(x, y) \sum b_j v_j)/Q(y) \in T(x, y)$ so that $Ra + Sb = \Theta$.

Case (c), the set of functions $q = (\sum a_j u_j - F(x, y) \sum b_j v_j)/Q(y) \in T(x, y)$ so that $a_j \geqslant 0$ if $j \in I$ and $x_j = 0$.

If $(\bar{x}, \bar{y}), (x, y) \in M$ then there is a $q \in T(x, y)$ so that

$$F(\bar{x}, \bar{y}) - F(x, y) = (Q(y)/Q(\bar{y}))q.$$

Since $Q(y)(t) > 0$ for every $t \in [0, 1]$ and $y \in N$ (F, M) has property S. Hence best approximations are characterized by the local Kolmogoroff condition. For case (b) that has already been proved in [7].

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